REMARKS ON A HILBERT SPACE OF ANALYTIC FUNCTIONS

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1. *Introduction.*—In a recent paper (hereafter quoted as *I*; see ref. 1), I studied a family of Hilbert spaces, $\mathfrak{F}_n$ $(n = 1, 2, \ldots)$, which may be described as follows: The elements of $\mathfrak{F}_n$ are entire analytic functions $f(z)$, where $z = (z_1, z_2, \ldots, z_n)$ is a point of the n-dimensional complex Euclidean space C_n. The inner product of two elements f, f' of $\mathfrak{F}_n$ is

$$\left.\begin{aligned} (f, f') &= \int_{C_n} \overline{f(z)} f'(z) d\mu_n(z) \\ d\mu_n(z) &= \pi^{-n} \exp\left(-\Sigma_{k=1}^n |z_k|^2\right) \Pi dx_k dy_k \qquad (z_k = x_k + iy_k), \end{aligned}\right\} \tag{1}$$

so that f belongs to $\mathfrak{F}_n$ if and only if $(f,f) < \infty$.

An orthonormal base of $\mathfrak{F}_n$ is given by

$$u_{[m]}(z) = \Pi_k \frac{z_k^{m_k}}{\sqrt{m_k!}} \tag{2}$$

where $m = (m_1, \ldots, m_n)$ denotes an n-tuple of nonnegative integers and k runs from 1 to n. Thus, every $f \in \mathfrak{F}_n$ may be written as

$$f(z) = \Sigma_m \gamma_{[m]} u_{[m]}(z), \tag{3}$$

and $\|f\|^2 = \Sigma |\gamma_{[m]}|^2$. For every $f \in \mathfrak{F}_n$, we have the inequality

$$|f(z)| \leqslant \|f\| e^{1(z,z)/2} \qquad (z,z) = \Sigma_k |z_k|^2. \tag{4}$$

A characteristic feature is the existence of a family of "principal vectors" $\mathbf{e}_a$ in $\mathfrak{F}_n$ such that for every $f \in \mathfrak{F}_n$ and $a \in C_n$,

$$f(a) = (\mathbf{e}_a, f). \tag{5}$$

The explicit form for $\mathbf{e}_a$ is

$$\mathbf{e}_a(z) = \exp(a, z) \qquad (a, z) = \Sigma_k \bar{a}_k z_k. \tag{6}$$

In *I*, §3d, the operators Y_j and Z_j $(1 \leqslant j \leqslant n)$ were introduced, viz.,

$$(Y_j f)(z) = \frac{\partial f(z)}{\partial z_j}, \qquad (Z_j f)(z) = z_j f(z). \tag{7}$$

Y_j and Z_j are adjoint (with respect to the inner product (1)), and satisfy the commutation rules,

$$[Y_j, Z_k] = \delta_{jk}, \qquad [Y_j, Y_k] = [Z_j, Z_k] = 0, \tag{7a}$$

of the annihilation-and-creation operators for "bosons" in quantum theory.

At the end of *I*, the case of a countably infinite family Y_j, Z_j was briefly considered, without, however, interpreting the Hilbert space on which they operate as a function space.

It is the purpose of this note to sketch the construction of such a space, $\mathfrak{F}_\infty$,

which may be considered a limiting case of the $\mathfrak{F}_n$, and to survey, briefly, some of its properties. A fuller and detailed account will appear in the *Communications on Pure and Applied Mathematics*.

2. *Definition of $\mathfrak{F}_\infty$.*—We first change the notation by letting

$$m = (m_1, m_2, \ldots)$$

stand for an *infinite* sequence of nonnegative integers of which at most a finite number are different from zero. The (countable) set of these sequences will be called M. The n-tuples used above are in a one-to-one correspondence with the elements of the subset M_n of M containing those m for which $m_k = 0$ if $k > n$.

Similarly, we introduce infinitely many complex variables, and set

$$z = (z_1, z_2, \ldots).$$

The functions $u_{[m]}(z)$ are defined as before, for all $m \in M$. (Since all but a finite number of m_k vanish, the product in equation (2) is finite.) For a set of coefficients $\gamma_{[m]}$, equation (3) defines a function $f(z)$ provided the series converges. It follows from the inequality (4) that the series in equation (3) converges absolutely if $\Sigma_m |\gamma_{[m]}|^2 < \infty$ as well as $\Sigma_k |z_k|^2 < \infty$. This suggests the following definition:

Let $\mathfrak{Z}$ be the Hilbert space of sequences z for which $\Sigma |z_k|^2 < \infty$, and with inner product $(z,z') = \Sigma_k \bar{z}_k z'_k$. $\mathfrak{F}_\infty$ is the Hilbert space of all functions

$$f(z) = \Sigma_m \in_M \gamma_{[m]} u_{[m]}(z) \qquad (z \in \mathfrak{Z})$$

with complex coefficients $\gamma_{[m]}$ satisfying $\Sigma_m |\gamma_{[m]}|^2 < \infty$. The inner product of two elements $f, f' \in \mathfrak{F}_\infty$ is

$$(f, f') = \Sigma_m \overline{\gamma_{[m]}} \gamma'_{[m]}. \tag{8}$$

For two elements f, g of $\mathfrak{F}_\infty$, $f = g$ will mean that $f(z) = g(z)$ for all z.

It is easily seen that f determines the coefficients $\gamma_{[m]}$ uniquely, and (8) is, therefore, a consistent definition.

The projections E_n: Let Q_n ($n = 1, 2, \ldots$) be the projection operator on $\mathfrak{Z}$ defined by

$$Q_n z = (z_1, z_2, \ldots, z_n, 0, 0, \ldots)$$

and introduce the projection E_n on $\mathfrak{F}_\infty$ by

$$(E_n f)(z) = f(Q_n z).$$

In particular $E_n u_{[m]} = u_{[m]}$ if $m \in M_n$, and it vanishes if $m \notin M_n$. $E_n \mathfrak{F}_\infty$ is isomorphic to $\mathfrak{F}_n$, and for every $f \in \mathfrak{F}_\infty$, $E_n f$ converges strongly to f as $n \to \infty$. This implies that for any two f, f' in $\mathfrak{F}_\infty$, $(f, f') = \lim_{n \to \infty} (E_n f, E_n f')$ or, by equation (1),

$$(f, f') = \lim_{n \to \infty} \int \overline{f(z_1, \ldots, z_n, 0, \ldots)} f'(z_1, \ldots, z_n, 0, \ldots) d\mu_n(z) \tag{9}$$

The inner product may thus be interpreted as the integral of $\bar{f}f'$ over the Hilbert space $\mathfrak{Z}$ (in Gaussian measure). For a general definition of such an integration process, see Segal[2] and Friedrichs-Shapiro.[3] We are concerned here with a severely restricted class of functions, so that the deeper and subtler problems of measure theory in Hilbert space are not even touched upon, and equation (9) is sufficient for our purpose.

3. *The Principal Vectors and the Linear Set* $\mathfrak{L}$.—If we define, for every $a \in \mathcal{B}$, $\mathbf{e}_a$ by equation (6), then $\mathbf{e}_a \in \mathfrak{F}_\infty$, and relation (5) holds for every $f \in \mathfrak{F}_\infty$. Setting $f = \mathbf{e}_b$, one obtains from (5)

$$(\mathbf{e}_a, \mathbf{e}_b) = \exp \; (b,a). \tag{10}$$

The inequality (4) is an immediate consequence of (5) and (10) via Schwarz' inequality.

$\mathfrak{L}$ is defined as the linear set of all *finite* linear combinations of principal vectors. By (5), $\mathfrak{L}$ is dense in $\mathfrak{F}_\infty$ because only the zero vector is orthogonal to all $\mathbf{e}_a$. Many results can, therefore, be easily established by verifying them for $\mathfrak{L}$, the extension to $\mathfrak{F}_\infty$ being straightforward. Thus, the following operators $V_a (a \in \mathcal{B})$ and V_S (S a unitary operator on $\mathcal{B}$) may be shown to be unitary on $\mathfrak{F}_\infty$:

$$(V_a f)(z) = f(z - a) \exp \left\{ (a,z) - \frac{1}{2}(a, a) \right\}; \tag{11}$$

$$(V_S f)(z) = f(S^{-1}z). \tag{12}$$

(They correspond to the operators introduced in *I*, §3a, and most of §3a may be literally taken over.)

The definition of $\mathfrak{F}_\infty$ has been given with reference to a fixed coordinate system in $\mathcal{B}$. The fact that V_S in (12) is a unitary operator on $\mathfrak{F}_\infty$ allows us to conclude, however, that the definition of $\mathfrak{F}_\infty$—including of course the definition of the inner product—is invariant under a unitary coordinate transformation in $\mathcal{B}$. (It would be quite appropriate to write $\mathfrak{F}[\mathcal{B}]$ instead of $\mathfrak{F}_\infty$.)

Alternative definition of $\mathfrak{F}[\mathcal{B}]$: Utilizing the linear set $\mathfrak{L}$, one may give a definition which, from the start, is independent of any coordinate system in $\mathcal{B}$. $\mathfrak{L}$ is made into a pre-Hilbert space by defining the inner product of two elements $f = \Sigma_{j=1}^{p} \lambda_j \mathbf{e}_{a_j}$, $f' = \Sigma_{k=1}^{q} \mu_k \mathbf{e}_{b_k}$ as

$$(f, f') = \Sigma_{j,k} \bar{\lambda}_j \mu_k \exp \; (b_k, a_j)$$

in accordance with equation (10). $\mathfrak{F}[\mathcal{B}]$ is then obtained by completion.

This procedure can also be applied to a nonseparable Hilbert space $\mathcal{B}$. The resulting $\mathfrak{F}[\mathcal{B}]$ differs relatively little from $\mathfrak{F}_\infty$, but in what follows we restrict ourselves to $\mathfrak{F}_\infty$.

Analytic properties of $f(z)$: It is not surprising that the functions $f(z)$ in $\mathfrak{F}_\infty$ turn out to be *analytic*—according to the definition of analytic functions on a normed vector space. (See Hille-Phillips,[4] chap. 3, sec. 3.) In view of equation (5), the principal vectors provide a convenient tool for studying the analytic properties of $f(z)$. Thus, one obtains, for a complex variable λ and two elements a, b of $\mathcal{B}$,

$$b \cdot \nabla f(a) = \frac{d}{d\lambda} f(a + \lambda b)|_{\lambda=0} = (h_b \mathbf{e}_a, f), \tag{13}$$

where $h_b(z) = (b,z)$. (Equation (13) is identical with the expression $\delta f(a;b)$ in Hille-Phillips.) One can also derive simple estimates such as

$$|b \cdot \nabla f(a)|^2 \leqslant \|b\|^2 (1 + \|a\|^2) \exp(\|a\|^2).$$

In $\mathfrak{F}_\infty$ there exist infinitely many operators of the form (7). Their precise definition is essentially the same as in *I*, §3d.

4. *Connection with the Fock Representation.*—(a) *Polynomials in* $\mathfrak{F}_\infty$: Set, for any $m \in M$, $|m| = \Sigma_k m_k$. $\mathfrak{F}_\infty$ may be decomposed into the direct sum of mutually orthogonal subspaces $\mathfrak{P}_g$ ($g = 0, 1, 2, \ldots$) where $\mathfrak{P}_g$ is spanned by all $u_{[m]}$ with $|m| = g$. The functions f belonging to $\mathfrak{P}_g$ are homogeneous "polynomials" of order g. (See reference 4, p. 760.) The principal vectors $\mathbf{e}_a^{(g)}$ of $\mathfrak{P}_g$ are the projections of $\mathbf{e}_a$ into $\mathfrak{P}_g$, viz., $\mathbf{e}_a^{(g)}(z) = (a,z)^g/g!$.

(b) *Fock space:* Let $\mathfrak{X}$ be the standard Hilbert space of one-particle quantum mechanics, i.e., the space of complex square integrable functions $\varphi(x)$. Here x is a point in three-dimensional real Euclidean space, and dx denotes Lebesgue measure. Furthermore, we introduce the g-particle spaces $\mathfrak{X}_g$ ($g = 2, 3, \ldots$) of complex square integrable symmetric functions $\varphi(x_1, \ldots, x_g)$ of g points, we set $\mathfrak{X}_1 = \mathfrak{X}$, and we write $\mathfrak{X}_0$ for the one-dimensional Hilbert space of constants (or complex numbers). (See, for example, Friedrichs,[5] §6.) The Fock space $\mathfrak{H}[\mathfrak{X}]$ is the direct sum of the $\mathfrak{X}_g$ ($g = 0, 1, 2, \ldots$). Its elements will be denoted by

$$\Phi = (\varphi_0, \varphi_1, \ldots, \varphi_g, \ldots) \qquad (\varphi_g \in \mathfrak{X}_g),$$

and the inner product of two elements Φ, Φ' is

$$(\Phi, \Phi') = \sum_{g=0}^\infty (\varphi_g, \varphi_g'),$$

where (φ_g, φ_g') is the inner product in $\mathfrak{X}_g$. (This implies that Φ belongs to $\mathfrak{H}[\mathfrak{X}]$ if $(\Phi, \Phi) < \infty$.) The subspace of the Φ for which all $\varphi_h = 0$ if $h \neq g$ will be identified with $\mathfrak{X}_g$.

(c) *Basic vectors:* Let $v_1(x), v_2(x), \ldots$ be an orthonormal basis of $\mathfrak{X}$. From a product $v = v_{i_1}(x_1)v_{i_2}(x_2) \ldots v_{i_g}(x_g)$ ($g \geqslant 1$) we obtain, by symmetrization and normalization, the function $w_{[m]}$, where $m \in M$, $|m| = g$, and m_k is the number of times the function v_k appears as a factor in v (the "occupation number" of the state v_k). If we set, in addition, $w_{[0]} = 1$, then $w_{[m]}$ with $|m| = g$ span $\mathfrak{X}_g$, and all $w_{[m]}$ form a basis for the Fock space $\mathfrak{H}[\mathfrak{X}]$. (See Dirac,[6] §59.)

(d) *Annihilation-and-creation operators:* It will be convenient to introduce $\mathfrak{X}^*$, the dual of $\mathfrak{X}$. $\mathfrak{X}^*$ is, of course, itself a Hilbert space of square integrable functions, say, $\zeta(x)$, and we write

$$\langle \zeta \varphi \rangle = \int \zeta(x)\varphi(x)dx.$$

We shall also set $\zeta^{(0)} = 1$, $\zeta^{(g)}(x_1, \ldots, x_g) = \prod_{i=1}^g \zeta(x_i)$, and write

$$\langle \zeta^{(0)} \varphi_0 \rangle = \varphi_0 \qquad \text{and}$$

$$\langle \zeta^{(g)} \varphi_g \rangle = \int \ldots \int \zeta^{(g)}(x_1, \ldots, x_g)\varphi_g(x_1, \ldots, x_g)dx_1 \ldots dx_g.$$

The basis of $\mathfrak{X}^*$ dual to v_i consists of the functions $\overline{v_i(x)}$.

For every $\beta \in \mathfrak{X}^*$, we define the operators

$$A(\beta) \equiv \int \beta(x)\Psi(x)dx \qquad \text{and} \qquad A(\beta)^* \equiv \int \overline{\beta(x)}\Psi^*(x)dx$$

as follows. Let $A(\beta)\Phi = \Phi'$, and $A(\beta)^*\Phi = \Phi''$, for a vector Φ in Fock space. Then,

$$\varphi_g'(x_1, \ldots, x_g) = (1 + g)^{1/2} \int \varphi_{g+1}(x_1, \ldots, x_g, x)\beta(x)dx; \qquad (14a)$$

$$\varphi_0'' = 0,$$
$$\varphi_g''(x_1, \ldots, x_g) = g^{-1/2}\Sigma_{\alpha=1}^{g}\varphi_{g-1}(x_1, \ldots, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_g)\overline{\beta(x_\alpha)} \quad\quad\Bigg\} \quad (14b)$$
$$(g \geqslant 1).\Bigg)$$

The functions φ_g' and φ_g'' belong to $\mathfrak{X}_g$, and Φ is in the domain of $A(\beta)$ (or $A(\beta)^*$) if and only if $\Sigma_{g=0}^{\infty}\|\varphi_g'\|^2 < \infty$ (or $\Sigma_{g=0}^{\infty}\|\varphi_g''\|^2 < \infty$).

(e) *Mapping of the Fock space onto* $\mathfrak{F}_\infty$: A unitary mapping D of $\mathfrak{H}[\mathfrak{X}]$ onto $\mathfrak{F}_\infty$ is defined by

$$Dw_{[m]} = u_{[m]} \quad\quad \text{(for all } m \in M). \quad\quad (15)$$

Clearly, $D\mathfrak{X}_g = \mathfrak{P}_g$. If $\Phi = \Sigma_m\gamma_{[m]}w_{[m]}$, then $f = D\Phi = \Sigma_m\gamma_{[m]}u_{[m]}$. In order to express $f(z)$ in terms of the φ_g, we introduce the following unitary mapping Δ of $\mathfrak{X}^*$ onto $\mathfrak{Z}$. Let $z = \Delta\zeta$; then,

$$z_i = \langle\zeta v_i\rangle \quad \text{or} \quad \zeta(x) = \Sigma_{i=1}^{\infty}z_i\overline{v_i(x)}. \quad\quad (15a)$$

It can be shown that with these definitions, for $f = D\Phi$,

$$f(z) = \Sigma_{g=0}^{\infty}(g!)^{-1/2}\langle\zeta^{(g)}\varphi_g\rangle \quad\quad (z = \Delta\zeta). \quad\quad (15b)$$

Since D is unitary, $\|f\|^2 = \Sigma_g\|\varphi_g\|^2$. (The series in (15b) converges absolutely for every z.)

It remains to determine the transforms of the annihilation and creation operators A and A^*. Let $b = \Delta\beta$, and $A'(b) = DA(\beta)D^{-1}$. If now $f' = A'(b)f$, and $f'' = (A'b)^*f$, then

$$f'(z) = b\cdot\nabla f(z), \quad f''(z) = (b,z)f(z). \quad\quad (16)$$

Again f belongs to the domain of A' (or $(A')^*$) if f' (or f'') belongs to $\mathfrak{F}_\infty$. (Incidentally, the domains of $A'(b)$ and $(A'(b))^*$ coincide.) The operators Y_j and Z_j of equation (7) are, respectively, the transforms of $A(\bar{v}_j)$ and $(A(\bar{v}_j))^*$.

The construction (15b) has already been developed by Fock (ref. 7, equation (5), p. 429). More precisely, our f corresponds to his Ω, our ζ to his $\bar{b}$. Similarly, our equation (16) corresponds to Fock's equations (13), p. 431, and (14), p. 432.

Friedrichs' procedure (ref. 5, §12) is also analogous to this construction, the main difference being that the mapping (15b) is onto a *complex* Hilbert space and onto analytic functions.

Remark: The precise nature of $\mathfrak{X}$ is irrelevant, and the mapping (15b) could just as well be defined for the more abstract formulation of Cook[8] and Segal.[2]

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ABSTRACT ERGODIC THEOREMS*

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1. Let $(Z, \mathcal{E}, \mu)$ be a complete totally σ-finite measure space and E a Banach space. For each $1 \leq p < \infty$ denote by $\mathcal{L}_E^p$ the vector space of all (Bochner) measurable mappings f of Z into E for which $z \to \|f(z)\|^p$ is μ-integrable; here $\mathcal{L}_E^p$ is endowed with the semi-norm $f \to \|f\|_p = (\int_Z \|f(z)\|^p d\mu(z))^{1/p}$. Denote by L_E^p the associated separated (Banach) space and by $f \to \tilde{f}$ the canonical mapping of $\mathcal{L}_E^p$ onto L_E^p. Let $\mathcal{S}_E$ be the vector space of all functions which are bounded and belong to $\mathcal{L}_E^1$; here $\mathcal{S}_E$ is endowed with the semi-norm $f \to \|f\|_\infty = \text{ess sup}_{z \in Z} \|f(z)\|$. Denote by S_E the associated separated (normed) space and by $f \to \tilde{f}$ the canonical mapping of $\mathcal{S}_E$ onto S_E.

Let $\mathfrak{D}$ be the set of all linear mappings T of S_E into S_E such that[1] $\|T\|_1 \leq 1$ and $\|T\|_\infty \leq 1$. Then $\|T\|_p \leq 1$ for all $1 \leq p < \infty$; hence, T can be extended by continuity to L_E^p (we denote the extension by the same letter). For $T \in \mathfrak{D}$ and $f \in \mathcal{v} = \bigcup_{1 \leq p < \infty} \mathcal{L}_E^p$ we denote by Tf a (determined) representative of the class $T\tilde{f}$.

2. Let $T_0, T_1, \ldots, T_k \in \mathfrak{D}$; consider the conditions:

(1) $T_0 = I$;

(2) $T_i T_j = T_j T_i$ for $i, j \in \{0, 1, \ldots, k\}$;

(3) $T_j T_{j+1} = T_{j+1}$ for $j \in \{0, 1, \ldots, k-1\}$.

We define $T_j^0 = I$ for all $j \in \{0, 1, \ldots, k\}$. For each function $f \in \mathcal{v}$ and each $a > 0$ let $G_f(a) = \{z \mid \|f(z)\| > a\}$.

THEOREM 1. *Let $T_0, T_1, \ldots, T_k \in \mathfrak{D}$ be $k + 1$ operators satisfying the conditions (1), (2), (3). For each $f \in \mathcal{v}$ and each $a > 0$, define*

$$G_f^*(a) = \{z \mid \sup_{j \in \{0,1,\ldots,k\}, n \in N} \|(T_j^0 + T_j^1 + \ldots + T_j^n)f(z)/(n+1)\| > a\}.$$

Then, for each set $F \in \mathcal{E}$ verifying (except for sets of measure zero) the relations $G_f(a) \subset F \subset G_f^*(a)$ we have

$$a\mu(F) \leq \int_F \|f(z)\| d\mu(z) < \infty.[2]$$

COROLLARY 1. *Let $T \in \mathfrak{D}$. For each $f \in \mathcal{v}$ and each $a > 0$, define*

$$E_f^*(a) = \{z \mid \sup_{n \in N} \|(T^0 + T^1 + \ldots + T^n)f(z)/(n+1)\| > a\}.$$

Then,

$$a\mu(E_f^*(a)) \leq \int_{E_f^*(a)} \|f(z)\| d\mu(z) < \infty.$$

Corollary 1 follows from Theorem 1 if we take $k = 1$, $T_0 = I$, $T_1 = T$ and $F = E_f^*(a)$.